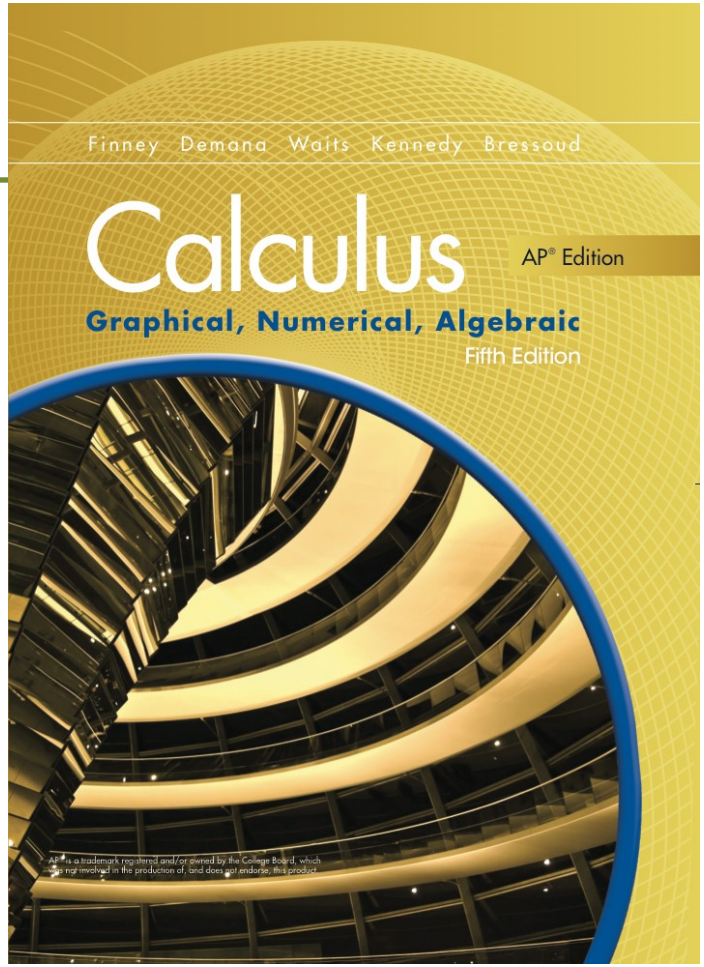
Chapter 5
Applications of Derivatives

Section 5.6

Related Rates



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Quick Review

1. Find the distance between the points (0,5) and (3,0).

Find dy/dx.

2. $2xy + y = x + 2$

3. $x \sin y = x + y$

4. $\ln(x + y) = 3x$

5. Let $x = 2\cos t,\ y = 2\sin t$. Find a parameter interval that produces the portion in the second and third quadrants, including the points on the axes.

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Quick Review Solutions

1. Find the distance between the points (0,5) and (3,0). $\sqrt{34}$

Find dy/dx.

2. $2xy + y = x + 2$ $\quad y' = \dfrac{1 - 2y}{2x + 1}$

3. $x \sin y = x + y$ $\quad y' = \dfrac{1 - \sin y}{x \cos y - 1}$

4. $\ln(x + y) = 3x$ $\quad y' = 3(x + y) - 1$

5. Let $x = 2\cos t$, $y = 2\sin t$. Find a parameter interval that produces the portion in the second and third quadrants, including the points on the axes. $\dfrac{\pi}{2} \le t \le \dfrac{3\pi}{2}$

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What you'll learn about

- Development of a mathematical model
- Creation of an equation that relates the variable whose rate of change is known to the variable whose rate of change is sought
- Use of Chain Rule to relate the rates of change
- Identification of solution
- Interpretation of solution

…and why

Related rate problems are at the heart of Newtonian mechanics; it was essentially to solve such problems that calculus was invented.

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Strategy for Solving Related Rate Problems

1. Understand the Problem In particular, identify the variable whose rate of change you seek and the variable (or variables) whose rate of change you know.

2. Develop a Mathematical Model of the Problem Draw pictures (many of these problems involve geometric figures) and label the parts that are important to the problem. Be sure to distinguish constant quantities from variables that change over time. Only constant quantities can be assigned numerical values at the start.

3. Write an equation relating the variable whose rate of change you seek with the variable(s) whose rate of change you know. The formula is often geometric, but it could come from a scientific application.

4. Differentiate both sides of the equation implicitly with respect to time. Be sure to follow all the differentiation rules. The Chain Rule will be especially critical, as you will be differentiating with respect to the parameter t.

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Strategy for Solving Related Rate Problems

5. Substitute values for any quantities that depend on time. Notice that it is only safe to do this after the differentiation step. Substituting too soon "freezes the picture" and makes changeable variables behave like constants, with zero derivatives.

6. Interpret the Solution Translate your mathematical result into the problem setting (with appropriate units) and decide whether the result makes sense.

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Example A Highway Chase

A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.8 mi north of the intersection and the car is 0.6 mi to the east, the police determine with radar that the distance between them and the car is increasing at 15 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

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Example A Highway Chase (cont'd)

Let x = the distance of the speeding car from the intersection.

Let y = the distance of the police car from the intersection.

Let z = the distance between the two cars.

We know that $dz/dt = 15$, $dy/dt = -60$, $x = 0.6$, $y = 0.8$, and $z = 1$.

Use the known information and the Pythagorean Theorem to find dx/dt.

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}z^2 \quad \Rightarrow \quad 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$2(0.6)\frac{dx}{dt} + 2(0.8)(-60) = 2(1)(15)$$

$$\frac{dx}{dt} = 105$$ The speeding car is travelling at 105 mph.

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Quick Quiz for Sections 5.4 – 5.6

You may use a graphing calculator to solve the following problems.

1. If Newton's method is used to approximate the real root of $x^3 + 2x - 1 = 0$, what would the third approximation, x_3, be if the first approximation is $x_1 = 1$?

(A) 0.453

(B) 0.465

(C) 0.495

(D) 0.600

(E) 1.977

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Quick Quiz for Sections 5.4 – 5.6

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1. If Newton's method is used to approximate the real root of $x^3 + 2x - 1 = 0$, what would the third approximation, x_3, be if the first approximation is $x_1 = 1$?

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(B) 0.465

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(D) 0.600

(E) 1.977

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Quick Quiz for Sections 5.4 – 5.6

2. The sides of a right triangle with legs x and y and hypotenuse z increase in such a way that $dz/dt = 1$ and $dx/dt = 3dy/dt$. At the instant when $x = 4$ and $y = 3$, what is dx/dt?

(A) 1/3

(B) 1

(C) 2

(D) $\sqrt{5}$

(E) 5

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Quick Quiz for Sections 5.4 – 5.6

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(A) 1/3

(B) 1

(C) 2

(D) $\sqrt{5}$

(E) 5

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Quick Quiz for Sections 5.4 – 5.6

3. An observer 70 meters south of a railroad crossing watches an eastbound train traveling at 60 meters per second. At how many meters per second is the train moving away from the observer 4 seconds after it passes through the intersection?

(A) 57.60

(B) 57.88

(C) 59.20

(D) 60.00

(E) 67.40

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Quick Quiz for Sections 5.4 – 5.6

3. An observer 70 meters south of a railroad crossing watches an eastbound train traveling at 60 meters per second. At how many meters per second is the train moving away from the observer 4 seconds after it passes through the intersection?

(A) 57.60

(B) 57.88

(C) 59.20

(D) 60.00

(E) 67.40

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Chapter Test

1. Given $f(x) = x^2 e^{1/x^2}$. Use analytic methods to find the intervals on which f is

(a) increasing

(b) decreasing

(c) concave up

(d) concave down

Then find any

(e) local extreme values

(f) inflection points

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Chapter Test

2. Use $y' = 6(x+1)(x-2)^2$ to find the points at which f has a

(a) local maximum

(b) local minimum

(c) point of inflection

3. Find the function with the derivative $f'(x) = \sin x + \cos x$ whose graph passes through the point $(\pi, 3)$.

4. Find the linearization $L(x)$ of $f(x) = \tan x$ at $x = -\pi/4$.

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Chapter Test

5. Let $f(x) = x \ln x$ on the interval $(a, b) = (0.5, 3)$.

(a) Find the value(s) of c in (a, b) for which $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

(b) Write an equation for the secant line AB where $A = (a, f(a))$ and $B = (b, f(b))$.

(c) Write an equation for the tangent line that is parallel to the secant line AB.

6. Let f be a function with $f'(x) = \sin x^2$ and $f(0) = -1$.

(a) Find the linearization of f at $x = 0$.

(b) Approximate the value of f at $x = 0.1$.

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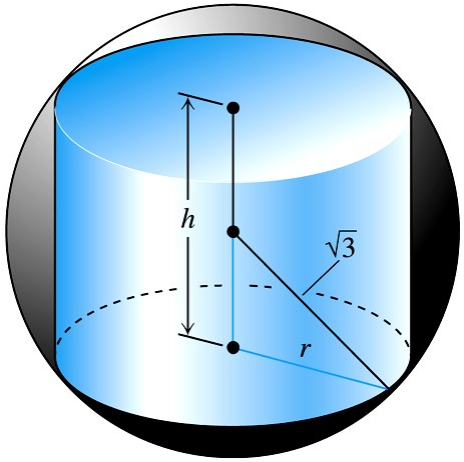
Chapter Test

7. Find the height and radius of the largest right circular cylinder that can be put into a sphere of radius $\sqrt{3}$ as described in the figure below.



8. The coordinates of a particle moving in the plane are differentiable functions of time t with $dx/dt = -1$ m/sec and $dy/dt = -5$ m/sec. How fast is the particle approaching the origin as it passes through the point (5, 12)?

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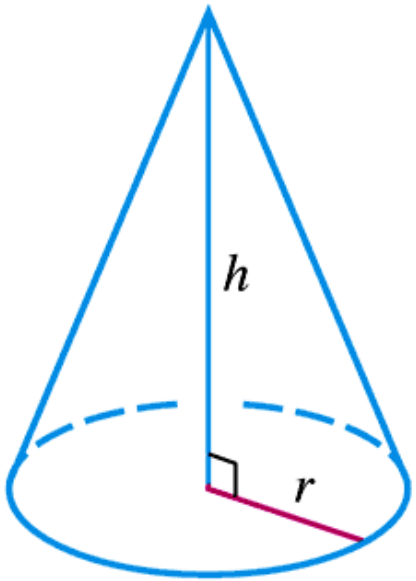
Chapter Test

9. Write a formula that estimates the change that occurs in the volume of a right circular cone when the radius changes from a to $a + dr$ and the height does not change.



$$V = \frac{1}{3}\pi r^2 h$$

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Chapter Test Solutions

1. Given $f(x) = x^2 e^{1/x^2}$. Use analytic methods to find the intervals on which f is

(a) increasing $[-1,0)$ and $[1,\infty)$

(b) decreasing $(-\infty,-1]$ and $(0,1]$

(c) concave up $(-\infty,0)$ and $(0,\infty)$

(d) concave down none

Then find any

(e) local extreme values at $x = -1$

(f) inflection points none

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Chapter Test Solutions

2. Use $y' = 6(x+1)(x-2)^2$ to find the points at which f has a

(a) local maximum none

(b) local minimum at $x = -1$

(c) point of inflection at $x = 0$ and $x = 2$

3. Find the function with the derivative $f'(x) = \sin x + \cos x$ whose graph passes through the point $(\pi, 3)$. $f(x) = -\cos x + \sin x + 2$

4. Find the linearization $L(x)$ of $f(x) = \tan x$ at $x = -\pi/4$.

$L(x) = 2x + \frac{\pi}{2} - 1$

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Chapter Test Solutions

5. Let $f(x) = x\ln x$ on the interval $(a, b) = (0.5, 3)$.

(a) Find the value(s) of c in (a, b) for which $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. $c \approx 1.579$

(b) Write an equation for the secant line AB where $A = (a, f(a))$ and $B = (b, f(b))$. $y \approx 1.457x - 1.075$

(c) Write an equation for the tangent line that is parallel to the secant line AB. $y \approx 1.457x - 1.579$

6. Let f be a function with $f'(x) = \sin x^2$ and $f(0) = -1$.

(a) Find the linearization of f at $x = 0$. $L(x) = -1$

(b) Approximate the value of f at $x = 0.1$. $f(0.1) \approx -1$

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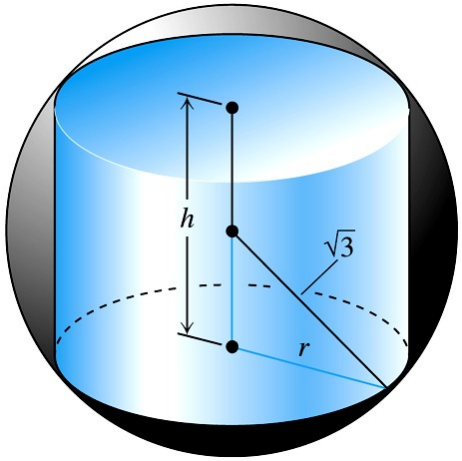
Chapter Test Solutions

7. Find the height and radius of the largest right circular cylinder that can be put into a sphere of radius $\sqrt{3}$ as described in the figure below.

Height $= 2$; Radius $= \sqrt{2}$



8. The coordinates of a particle moving in the plane are differentiable functions of time t with $dx/dt = -1$ m/sec and $dy/dt = -5$ m/sec. How fast is the particle approaching the origin as it passes through the point (5, 12)? 5 m/sec

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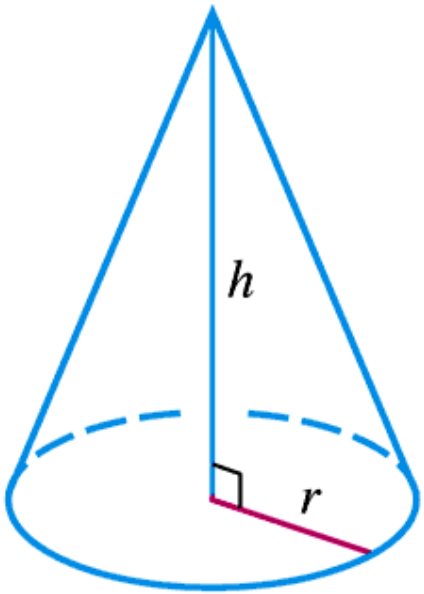
Chapter Test Solutions

9. Write a formula that estimates the change that occurs in the volume of a right circular cone when the radius changes from a to $a + dr$ and the height does not change. $dV \approx \frac{2\pi ah}{3} dr$



$$V = \frac{1}{3}\pi r^2 h$$

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